# STABILITY ROBUSTNESS OF CONTROL SYSTEMS OF MULTIROTOR UNMANNED AERIAL VEHICLES

**https://doi.org /10.59982/18294359-24.15-sv-19**

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***Abstract***

Multirotor unmanned aerial vehicles (UAVs) are broadly used in various military and civilian areas and tasks. In real flights of the multirotor UAVs, certain unexpected situations may occur, which can lead to improper functionality and even failures of elements or devices of the UAV's control system. In some cases, this can even lead to the complete collapse of the UAV. Therefore, the safety of UAV flights is nowadays of prime importance.

A method of analysis of the stability robustness of the UAV's control systems with respect to the parameters multiplicative uncertainties (or perturbations), including possible efficiency degradation of DC motors, is proposed in the paper. Stability robustness is determined by a simple graphical procedure on the complex plane of the Nyquist or Nichols hodographs of the control system's separate channels. That procedure is quite similar to the well-known classical feedback control procedure of determining the peak gain of the single-input, single-output closed-loop control system by the Nyquist hodograph of the open-loop system. A numerical example illustrating the proposed method is given.

***Keywords:*** Multirotor UAV, stability robustness, multiplicative uncertainty, motors efficiency degradation, multivariable control system.

## *Introduction*

Multicopters, or $N$ -rotor copters, also called multirotor unmanned aerial vehicles (UAV), are widely used in various military, search and rescue, and other civilian fields [Hassanalian, Abdelkefi, 99-120]. In real flights of the UAVs some unexpected situations may occur bringing to failures or changes in the characteristics of various elements or devices of the UAV's control system. This, in turn, can even lead to the complete collapse of the aerial vehicle. Therefore, the safety of the UAVs in case of uncertainties in control system elements are nowadays of prime importance.

That is why the so-called fault-tolerant control systems of UAVs have attracted much interest among researchers in recent years [Yushu, Yiqun, 1-10]. Many advanced control methodologies have been proposed to overcome the problem of elements' failures, including optimal control, model predictive control, sliding mode control and some others. Most of these methodologies result in complicated technical solutions and are rarely used in practice.

Another widely used and effective approach to solving the problem is based on the methods of robust control [Green, Limebeer, 35-50], [Kwakernaak, 255-273]. These methods allow one to develop systems that are rather simple in practical realization but can tolerate possible uncertainties in some elements of the UAV's control system.

A method of analysis of robustness of UAV's control systems with respect to multiplicative uncertainties, including possible partial efficiency degradation of motors, is proposed in the paper. The ultimately allowable norm of the multiplicative perturbation is determined by a simple graphical procedure on the complex plane of the Nyquist or Nichols hodographs of the system's separate channels. A numerical example illustrating the proposed method of analysis of the UAV's control system robustness is given.

***Rigid-Body Dynamics and Control System of Multirotor UAVs***

Let us denote $m$ the mass of the UAV; $g$ - gravitational constant; $J$ - inertia tensor of the UAV; $\omega = [\omega_x, \omega_y, \omega_z]^T$ - vector of angular velocities; $J_R$ - identical inertias of $N$ rotors; $\Omega_i$ $(i = 1, 2, \ldots, N)$ - angular velocities of the rotors; $\psi$ , $\phi$ , $\theta$ - yaw, roll and pitch angles.

Then the nonlinear equations of motion of the *N*-rotor UAV can be written in the form:

$$m\frac{d^2\xi}{dt^2} = -mgz_I + RF , \tag{1}$$

$$J\frac{d\omega}{dt} + \omega \times \left(J\omega + \Upsilon_R \Omega\right) = \tau , \tag{2}$$

$$\frac{d\eta}{dt} = P(\eta)\omega , \tag{3}$$

where $R$ is an orthogonal rotation matrix [Mahony et al., 23], $\Upsilon_R = [0 \;\; 0 \;\; J_R]^T$ , $\Omega$ denotes the total angular velocity of the rotors, and the matrix $P(\eta)$ in the equation (3) is equal to

$$P(\eta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta \ \mathrm{tg}\phi & 1 & \cos\theta\,\mathrm{tg}\phi \\ \sin\theta / \cos\phi & 0 & \cos\theta / \cos\phi \end{bmatrix}. \tag{4}$$

The vectors $F$ , $\tau = [\tau_x, \tau_y, \tau_z]^T$ in the equations (1), (2) define the principal non-conservative forces and moments applied to the UAV by the $N$ rotors. Each $i$ -th rotor generates a thrust $T_i$ , which is proportional to the square of angular velocity $\Omega_i$ . Denoting the total thrust by $T_\Sigma$ $(T_\Sigma = \sum_{i=1}^{N} T_i)$ , and by $\bar{T}$ , the $N$ -dimensional vector of thrusts $T_i$ ($\bar{T} = [T_1, T_2, \ldots, T_N]^T$), the mapping of $\bar{T}$ to the vector $[T_\Sigma, \tau]^T$ can be written, generally, in matrix form

$$\begin{bmatrix} T_\Sigma \\ \tau \end{bmatrix} = D_M \Lambda_M \bar{T} , \quad \Lambda_M = diag\left\{\lambda_i^M\right\} , \tag{5}$$

where the $4 \times N$ full-rank numerical matrix $D_M$ depends on the UAV geometry, number of rotors $N$ , etc. [Mahony et al., ], and $\lambda_i^M$ $(0 < \lambda_i^M \le 1)$ are the motors' degradation parameters. For properly functioning motors, the matrix $\Lambda_M$ is equal to the identity matrix $I$ . Given the needed controls $T_\Sigma$ and $\tau$ , the equation (5) allows computing the required thrusts $T_i$ of rotors.

Irrespective of the number of rotors $N$ , the flight altitude $z$ and the vector of rotations $\eta = [\phi, \theta, \psi]^T$ are chosen as four control variables of the UAVs, where the motion along the vertical inertial axis $z_I$ is described by the following scalar equation:

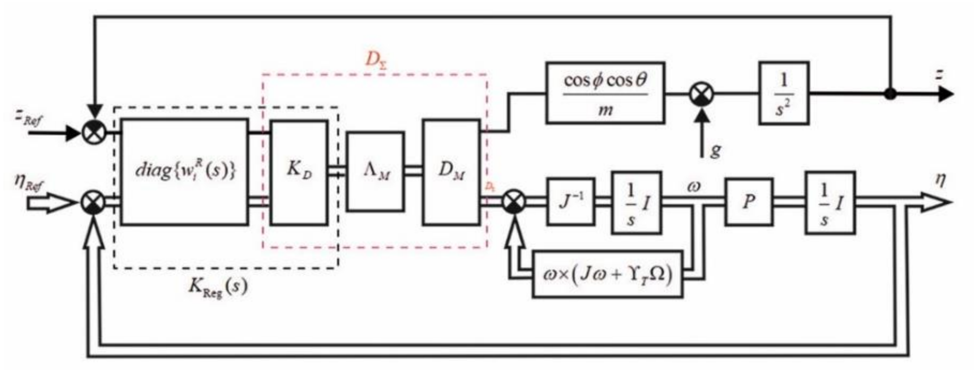
$$m\frac{d^2z}{dt^2} = (\cos\phi\cos\theta)u_z - mg, \quad (u_z = T_\Sigma). \tag{6}$$

or

$$\frac{d^2z}{dt^2} = \frac{\cos\phi\cos\theta}{m}u_z - g$$

The block diagram of the UAV's nonlinear control system can schematically be depicted in the form presented in Fig. 1 [Gasparyan et al., 60-63]. The scalar signals in the block diagram in Fig. 1 correspond to the vertical motion of the UAV, the double lines designate vectors of appropriate dimensions, and $s$ is the Laplace operator.

***Figure 1***




***Matrix block diagram of the UAV control system***

The system in Fig. 1 belongs to multi-input multi-output (MIMO) feedback control systems [Gasparyan, 14-35]. Usually, the matrix regulator $K_{\mathrm{Reg}}(s)$ in such systems is taken in the form

$$K_{\mathrm{Reg}}(s) = K_D diag\{w_i^R(s)\} . \tag{7}$$

In (7), $K_D = D_M^{-1}$ for $N = 4$, and $K_D = D_M^+$ for $N = 6$ or $N = 8$, where $D_M^+$ is the Moore-Penrose pseudoinverse of $D_M$, and $w_i^R(s)$ are the scalar transfer functions of the regulators in the separate channels. In practice, the standard PID regulators are often used as $w_i^R(s)$ in (7).

Let us denote $D_\Sigma = \{d_{ij}^\Sigma\}$ the following matrix:

$$D_\Sigma = D_M \Lambda_M K_D = D_M \Lambda_M D_M^+ . \tag{8}$$

In case of no motors' degradations, we have $D_\Sigma = I$ for any $N$, i.e. the cross-connections between four separate channels of the system in Fig. 1 are compensated. For that reason, the regulator $K_{\mathrm{Reg}}(s)$ (7) is called decoupling regulator [Gasparyan, 152-156].

In what follows, we shall assume that the angles and angular velocities of the UAV are so small that the nonlinear terms in the equations of rotational motions (2) can be neglected, and the cosines of all angles are approximately equal to unity. On these conditions, the dynamics equations (1),(2) take on the following linearized form:

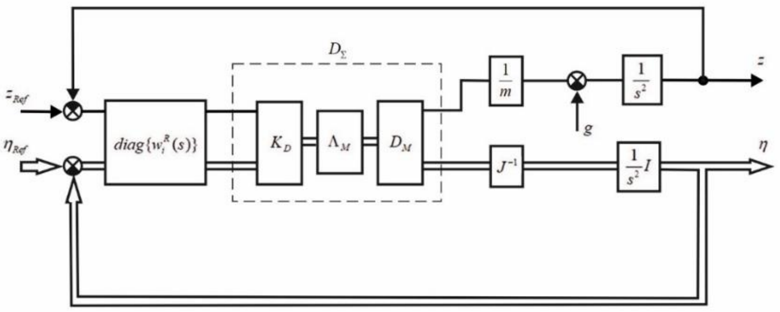
$$\frac{d^2z}{dt^2} = \frac{1}{m}T_\Sigma - g \ , \tag{9}$$

$$J\frac{d\omega}{dt} = \tau \ , \tag{10}$$

and the matrix block diagram of the control system in Fig. 1 reduces to the form shown in Fig. 2.

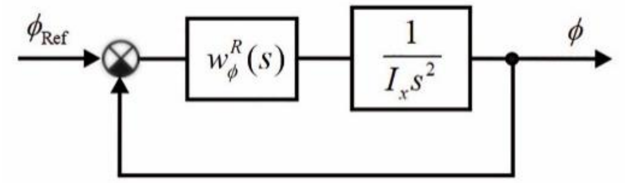
If $\Lambda_M = I$ and $K_D = D_M^+$, then all kinematic cross-couplings between separate channels of the linearized MIMO control system in Fig. 2 are compensated and the system splits into four independent single-input single-output (SISO) linear systems. As an instance, the roll channel $\phi$ of the decoupled MIMO control system in Fig. 2 is shown in Fig. 3, where $I_x$ is the moment of inertia around the roll axis. On the other hand, in case of motors' partial degradations, i.e. for $\Lambda_M \neq I$, the linearized MIMO system in Fig. 2 has cross-connections between separate channels.

***Figure 2***




***Matrix block diagram of the linearized control system of the UAV***

***Figure 3***



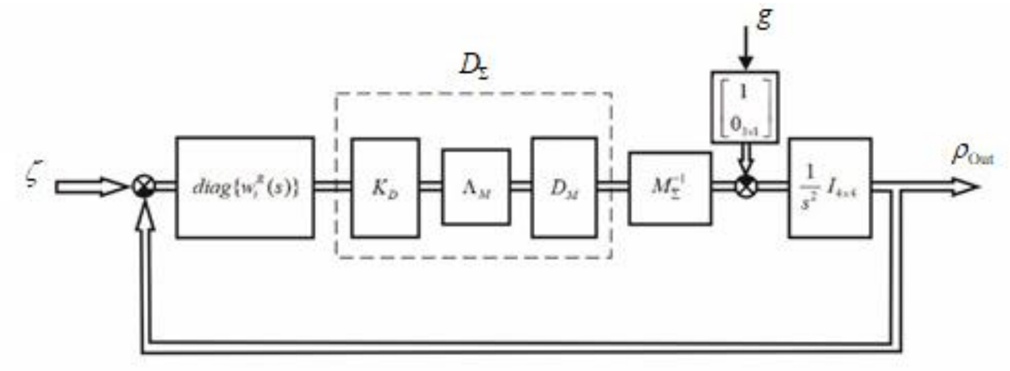

***Block diagram of the decoupled linear control system (roll channel)***

To analyze the stability robustness of the UAV's control system, it is more appropriate to transform the block diagram in Fig. 2 to the equivalent four-dimensional case in Fig. 4, where the vectors $\zeta(s)$, $\rho_{\text{Out}}(s)$ of size 4x1 and the 4x4 diagonal matrix $M$ are given by the following expressions:

$$\zeta(s) = \begin{bmatrix} z_{\text{Ref}}(s) \\ \eta_{\text{Ref}}(s) \end{bmatrix}, \quad \rho_{\text{Out}} = \begin{bmatrix} z(s) \\ \eta(s) \end{bmatrix}, \quad M_\Sigma = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_x & 0 & 0 \\ 0 & 0 & I_y & 0 \\ 0 & 0 & 0 & I_z \end{bmatrix}, \tag{11}$$

in which the vector $\eta$ in the vector $\rho_{\text{Out}}(s)$ is equal to $\eta = [\phi, \theta, \psi]^T$.

***Figure 4***




***Transformed block diagram of the linear control system of the UAV***

The transfer matrix of the open-loop MIMO system in Fig. 4 in case of $\Lambda_M \neq I$ has the form:

$$W(s) = \frac{1}{s^2} M_\Sigma^{-1} D_\Sigma diag\left\{w_i^R(s)\right\}. \tag{12}$$

Below, we shall admit for simplicity that all regulators $w_i^R(s)$ in (7) are identical, that is $w_i^R(s) = w_R(s)$. Then, instead of (12), we have

$$W(s) = \frac{1}{s^2} w_R(s) M_\Sigma^{-1} D_\Sigma. \tag{13}$$

The transfer matrix $\Phi(s)$ of the closed-loop control system of the UAV in Fig. 4 with respect to the output signals is equal to

$$\Phi(s) = \left[I + W(s)\right]^{-1} W(s), \tag{14}$$

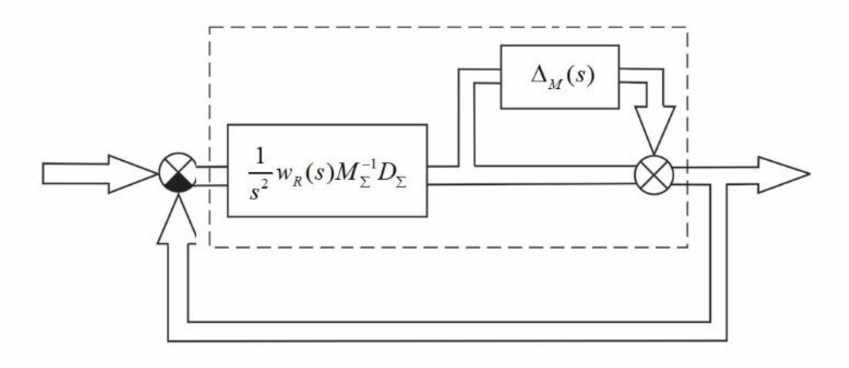
and the stability of the closed-loop system is determined by the roots of the characteristic equation:

$$\det[I + W(s)] = 0. \tag{15}$$

Note, that in case $\Lambda_M = I$, the matrix $D_\Sigma$ in (13) equals $I$ and the transfer matrix $\Phi(s)$ (14) of the closed-loop system takes on a diagonal form [Gasparyan et al., 65]. In other words, if $\Lambda_M = I$, then the stability of the UAV's control system is determined by stability of independent separate channels.

Let us discuss now the robustness of the UAV's control system with respect to any multiplicative uncertainties (perturbations) in the transfer matrix $W(s)$ (13), including possible partial degradations of the motor's efficiency. The corresponding matrix block diagram of the UAV's control system is shown in Fig. 5, where all uncertainties are absorbed in $\Lambda_M(s)$.

***Figure 5***




***UAV's control system with multiplicative uncertainties***

Based on the general robust theory [Green, Limebeer, 35-50], [Kwakernaak, 255-273] ,it can be shown that the robustness condition of the UAV's control system with respect to the multiplicative uncertainty $\Delta_M(s)$ can be written in the following form:

$$\|\Phi(j\omega)\|_\infty \le \frac{1}{\|\Delta_M(j\omega)\|_\infty}, \tag{16}$$

where the so-called Hardy norm $\|\Phi(j\omega)\|_\infty$ in (16) is determined as the strict upper bound of the largest singular value (denoted as $\bar{\sigma}$) of the transfer matrix $\Phi(j\omega)$ (14) over the whole frequency range of $\omega$ $(0 \le \omega \le \infty)$ , and is equal to:

$$\|\Phi(j\omega)\|_\infty = \sup_\omega \bar{\sigma}\left(\Phi(j\omega)\right). \tag{17}$$

The Hardy norm $\|\Delta_M(j\omega)\|_\infty$ of the perturbation matrix $\Delta_M(j\omega)$ is determined analogously. Below, for simplicity of notation, we shall wright just $\|\Delta_M\|$ instead of $\|\Delta_M(j\omega)\|_\infty$

It is easy to notice that for the diagonal transfer matrix $\Phi(s)$ (14), the largest singular value $\bar{\sigma}$ at any frequency $\omega$ is determined as the largest of the absolute values of the diagonal elements of $\Phi(j\omega)$. This allows imparting a simple geometrical interpretation to the robust condition (16). Let us re-wright the condition (16), accounting for (13) and (14) and the above remark, in the form:

$$\sup_\omega \left[ \max_i \left| \frac{w_i(j\omega)}{1 + w_i(j\omega)} \right| \right] \le \frac{1}{\|\Delta_M\|}, \tag{18}$$

where $w_i(j\omega)$ $(i = 1, 2, 3, 4)$ denote the transfer functions of the separate channels of the open-loop control system of the UAV in case of $\Lambda_M = I$ (e.g., $w_1(j\omega) = w(j\omega)/m$, $w_2(j\omega) = w(j\omega)/I_x$, etc.).

Then, to get the numerical estimates of allowable multiplicative perturbations based on the condition (18), the well-known in the classical feedback control graphical procedure of determining the peak gain of the SISO control systems can be used [Dorf, Bishop, 357]. It can be shown that if we pass in (18) to the equality sign, then that condition for any $i$ is reduced to the form:

$$\left[ \mathrm{Re}\{w_i(j\omega)\} + \frac{1}{1 - \|\Delta_M\|^2} \right]^2 + \left[\mathrm{Im}\{w_i(j\omega)\}\right]^2 = \frac{\|\Delta_M\|^2}{(1 - \|\Delta_M\|^2)^2}. \tag{19}$$

Geometrically, this expression determines on the complex plane of the Nyquist hodograph $w_i(j\omega)$ a circle with the center at the real point $-1/(1 - \|\Delta_M\|^2)$ and the radius equal to

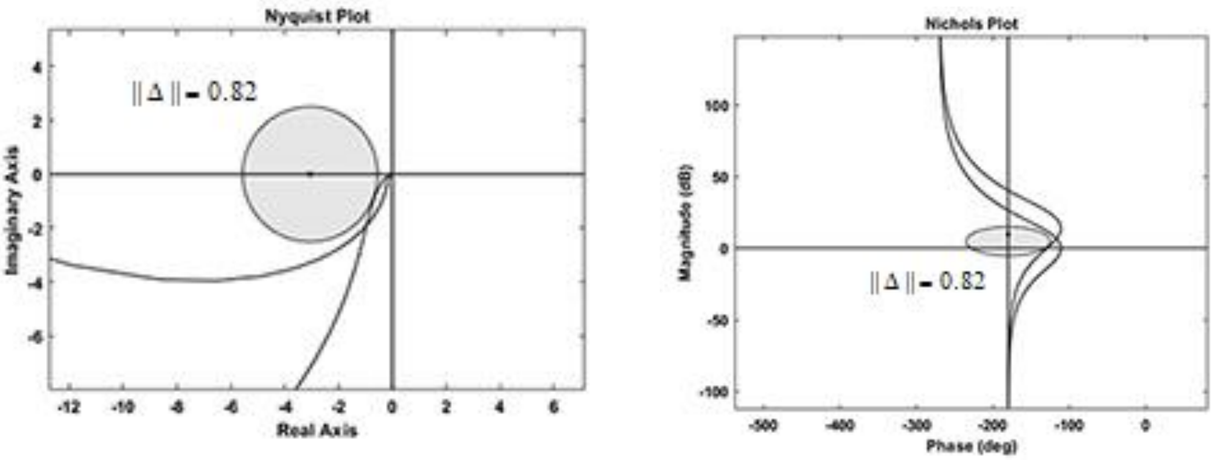
$r = \|\Delta_M\| / (1 - \|\Delta_M\|^2)$. The allowable value $\|\Delta_M\|$ is determined by the minimal radius of the circle, which touches the hodographs $w_i(j\omega)$ for all $i$ ($i = 1, 2, 3, 4$). Note that for $\|\Delta_M\| \to 0$, the circle (19) shrinks to the critical point $\{-1, j0\}$. Note that, by definition, the norm $\|\Delta_M\|$ is always less than unity.

***Numerical example.*** Consider a control system of a quadrotor with the following parameters: $m = 2.5\ kg$, $I_x = I_y = I_z = 0.5\ kg \cdot m^2$, and identical PID-regulators having the transfer function:

$$w_R(s) = 0.4642 + \frac{0.02173}{s} + \frac{2.435}{0.1834s + 1}. \tag{20}$$

In Fig. 6, there are shown the Nyquist and Nichols hodographs $w_1(j\omega)$ and $w_2(j\omega) = w_3(j\omega) = w_4(j\omega)$ of the separate channels of the UAV's control system with the PID-regulators (20) and the circle (area in the Nichols plot) tangent to the hodographs $w_{2,3,4}(j\omega)$. Based on (19), this circle determines the largest allowable value of the multiplicative uncertainty equal to $\|\Delta_M\| = 0.82$. Larger values of $\|\Delta_M\|$ may result in loss of stability of the UAV's control system.

***Figure 6***




*a) Nyquist plots* *b) Nichols plots*

***Stability robustness analysis of the UAV' control system***

***Conclusion***

A method of analysis of robustness of the UAV's control system with respect to the multiplicative uncertainties, including possible losses of DC motors' efficiency, is proposed in the paper. The largest allowable (by the Hardy norm) uncertainty is determined by a simple graphical procedure on the complex plane of Nyquist or Nichols hodographs of separate channels of the UAV's control system. A numerical example illustrating the proposed method of stability robustness analysis is given.

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Ներկայացվել է՝ 15.04.2024թ.
Ուղարկվել է գրախոսման՝ 10.05.2024թ.